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Directed strongly regular graphs from $1\frac{1}{2}$-designs

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ABSTRACT

Some families of directed strongly regular graphs with $t = \mu$ are constructed by using antiflags of $1\frac{1}{2}$-designs.

1. Introduction

A *finite incidence structure* consists of a finite set P of *points*, a set $\mathcal{B}$ of *blocks*, and an *incidence relation* $\in$ between points and blocks. An incident point–block pair is called a *flag*, and a non-incident point–block pair is called an *antiflag*. A *tactical configuration with parameters* $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$ is a finite incidence structure $\mathcal{T} = (P, \mathcal{B}, \in)$ with $|P| = \mathbf{v}$, $|\mathcal{B}| = \mathbf{b}$ such that every block contains $\mathbf{k}$ points and every point belongs to exactly $\mathbf{r}$ blocks.

A $1\frac{1}{2}$-*design* [6] or *partial geometric design* [1] with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$ is a tactical configuration $\mathcal{T} = (P, \mathcal{B}, \in)$ with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$ satisfying the following property.

> For every point $x \in P$ and every block $B \in \mathcal{B}$, the number of flags (y, C) such that $y \in B$ and $C \ni x$ is a if $x \notin B$ and b if $x \in B$.

Examples of $1\frac{1}{2}$-designs include 2-designs, complete bipartite graphs $K_{n,n}$, transversal designs, and partial geometries. The dual of a $1\frac{1}{2}$-design is again a $1\frac{1}{2}$-design (see [6]).

A *directed strongly regular graph* [4] with parameters (v, k, t, λ, μ) is a directed graph Γ on v vertices without loops such that (i) every vertex has in-degree and out-degree k, (ii) every vertex x has t out-neighbors that are also in-neighbors of x, and (iii) the number of directed paths of length 2 from a vertex x to another vertex y is λ if there is an edge from x to y, and is μ if there is no edge from x to y. Such a graph Γ is called a DSRG(v, k, t, λ, μ).

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Let I denote the identity matrix, and J the all-1 matrix (not necessarily square), with sizes that are clear from the context. The adjacency matrix of a directed strongly regular graph is a square $(0, 1)$-matrix A with zero diagonal such that the $\mathbb{Z}$-linear span of I, J, and A is closed under matrix multiplication. Equivalently, it is a square $(0, 1)$-matrix A with zero diagonal such that for certain constants k, t, λ, μ we have $AJ = JA = kJ$ and $A^2 = tI + \lambda A + \mu(J - I - A)$.

The incidence matrix of a $1\frac{1}{2}$-design is a $(0, 1)$-matrix N such that for certain constants $\mathbf{k}, \mathbf{r}, a, b$ we have $JN = \mathbf{k}J$, $NJ = \mathbf{r}J$, and $NN^\top N = (b - a)N + aJ$.

In this note, we observe the following. Given a $1\frac{1}{2}$-design with incidence matrix N, define a matrix A, with rows and columns indexed by the point–block pairs (p, B) for which $N_{pB} = 0$, by $A_{(p,B),(q,C)} = N_{pC}$. Then A is a directed strongly regular graph. This yields directed strongly regular graphs with previously unknown parameters (see [2,3]).

2. Construction

We show that the set of antiflags of a $1\frac{1}{2}$-design gives rise to a directed strongly regular graph with parameters $t = \mu$.

Theorem 2.1. *Let $\mathcal{T} = (P, \mathcal{B}, \in)$ be a tactical configuration with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$. Let $\Gamma = \Gamma(\mathcal{T})$ be the directed graph defined by*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : p \notin B\}$$

and

$$(p, B) \to (q, C) \quad \text{if and only if} \quad p \in C.$$

Then Γ is directed strongly regular if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof. Let Γ have adjacency matrix A. Write pB for an antiflag (p, B). Then

$$(A^2)_{pB,qC} = \sum_{rD} A_{pB,rD}A_{rD,qC} = \sum_{r,D} N_{pD}N_{rC}(1 - N_{rD}) = \mathbf{kr} - (NN^\top N)_{pC}.$$

If $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$, then $NN^\top N = (b - a)N + aJ$, and hence $A^2 = (\mathbf{kr} - a)J - (b - a)A$, so Γ is a directed strongly regular graph with parameters

$$v = \mathbf{b}(\mathbf{v} - \mathbf{k}), \qquad k = \mathbf{r}(\mathbf{v} - \mathbf{k}), \qquad t = \mu = \mathbf{kr} - a, \qquad \lambda = \mathbf{kr} - b.$$

Conversely, suppose that Γ is a DSRG(v, k, t, λ, μ). Then $A^2 = (t - \mu)I + (\lambda - \mu)A + \mu J$, and we find that $\mathbf{kr} - (NN^\top N)_{pC} = (t - \mu)\delta_{pB,qC} + (\lambda - \mu)N_{pC} + \mu$ for all antiflags pB, qC (where $\delta_{pB,qC}$ is 1 when $pB = qC$ and 0 otherwise). If $t \neq \mu$ then $\delta_{pB,qC}$ is determined by p, C and is independent of q, B. This can hold only for $\mathbf{v} = \mathbf{k} + 1$, $\mathbf{b} = \mathbf{r} + 1$, and $A = J - I$, so μ is undefined. Therefore, we may assume that $t = \mu$, so $NN^\top N = (\mu - \lambda)N + (\mathbf{kr} - \mu)J$. $\square$

Similarly, the set of flags of a $1\frac{1}{2}$-design gives a directed strongly regular graph with $t = \lambda + 1$.

Theorem 2.2. *Let $\mathcal{T} = (P, \mathcal{B}, \in)$ be a tactical configuration with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r})$. Let Γ be the directed graph defined by*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : p \in B\}$$

and

$$(p, B) \to (q, C) \quad \text{if and only if} \quad (p, B) \neq (q, C) \text{ and } p \in C.$$

Then Γ is directed strongly regular if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof. This time, write pB for a flag (p, B). Let Γ have adjacency matrix A, and put $M = A + I$ so that $M_{pB,qC} = N_{pC}$. Then

$$(M^2)_{pB,qC} = \sum_{rD} M_{pB,rD} M_{rD,qC} = \sum_{r,D} N_{pD} N_{rC} N_{rD} = (NN^\top N)_{pC}.$$

If $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b)$, then $NN^\top N = (b-a)N + aJ$, and $M^2 = (b-a)M + aJ$, so $A^2 = (M-I)^2 = (b-a-2)A + (b-a-1)I + aJ$, and it follows that Γ is a directed strongly regular graph with parameters

$$v = \mathbf{vr}, \qquad k = \mathbf{rk} - 1, \qquad t = b - 1, \qquad \lambda = b - 2, \qquad \mu = a.$$

Conversely, suppose that Γ is a DSRG(v, k, t, λ, μ). Then $A^2 = (t-\mu)I + (\lambda-\mu)A + \mu J$, so $M^2 = (\lambda - \mu + 2)M + \mu J + (t - \lambda - 1)I$, and therefore $(NN^\top N)_{pC} = (\lambda - \mu + 2)N_{pC} + \mu + (t - \lambda - 1)\delta_{pB,qC}$. If $t \neq \lambda + 1$, then $\delta_{pB,qC}$ is determined by p, C and is independent of q, B. This can hold only for $\mathbf{k} \leq 1$, $\mathbf{r} \leq 1$, and λ is undefined. Therefore, we may assume that $t = \lambda + 1$, so $NN^\top N = (\lambda - \mu + 2)N + \mu J$. $\square$

3. Examples

In this section, we give some concrete examples of new directed strongly regular graphs that are constructed by Theorem 2.1.

Example 3.1. Let P be the set of $2n$ vertices, and let $\mathcal{B}$ be the set of n^2 edges of the complete bipartite graph $K_{n,n}$. Then the incidence structure $\mathcal{T} = (P, \mathcal{B}, \in)$ is a $1\frac{1}{2}$-design with parameters

$$\mathbf{v} = 2n, \qquad \mathbf{b} = n^2, \qquad \mathbf{k} = 2, \qquad \mathbf{r} = n; \qquad a = 1, \qquad b = n + 1.$$

Therefore the graph $\Gamma(\mathcal{T})$ is a directed strongly regular graph with parameters

$$v = 2n^2(n-1), \qquad k = 2n(n-1), \qquad t = \mu = 2n - 1, \qquad \lambda = n - 1.$$

For $n = 3, 4$ we obtain directed strongly regular graphs with new parameter sets (36, 12, 5, 2, 5) and (96, 24, 7, 3, 7). By Duval [4], if there exists a DSRG(v, k, t, λ, μ) with $t = \mu$, then there also are DSRG$(mv, mk, mt, m\lambda, m\mu)$ for all positive integers m. In particular, we also find directed strongly regular graphs with parameter sets (72, 24, 10, 4, 10) and (108, 36, 15, 6, 15).

Example 3.2. A partial geometry $pg(\kappa, \rho, \tau)$ is a set of points P, a set of lines $\mathcal{L}$, and an incidence relation between P and $\mathcal{L}$ with the following properties.

(1) Every line is incident with κ points ($\kappa \geq 2$), and every point is incident with ρ lines ($\rho \geq 2$).
(2) Any two points are incident with at most one line.
(3) If a point p and a line L are not incident, then there exist exactly τ ($\tau \geq 1$) lines that are incident with p and meet L.

A partial geometry $pg(\kappa, \rho, \tau)$ is a $1\frac{1}{2}$-design $\mathcal{T}$ with parameters

$$(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (\kappa c,\ \rho c,\ \kappa,\ \rho;\ \tau,\ \mathbf{r} + \mathbf{k} - 1),$$

where $c = 1 + (\kappa - 1)(\rho - 1)/\tau$.

For example, the partial geometry obtained from an affine plane of order q by considering all q^2 points and taking the lines of l parallel classes is a $pg(q, l, l-1)$ and hence yields a $1\frac{1}{2}$-design $\mathcal{T}$ with parameters $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (q^2, ql, q, l; l-1, q+l-1)$ and a directed strongly regular graph $\Gamma(\mathcal{T})$ defined as in Theorem 2.1 with parameters

$$(v, k, t, \lambda, \mu) = (lq^2(q-1),\ lq(q-1),\ lq - l + 1,\ (l-1)(q-1),\ lq - l + 1).$$

For example, for $q = l = 3$, we find the previously unknown graph with parameter set (54, 18, 7, 4, 7). Doubling yields the graph with parameter set (108, 36, 14, 8, 14).

Remark 3.3. The above characterization theorems may be used to show non-existence of $1\frac{1}{2}$-designs with given parameter sets. We give one example. Suppose that there exists a $1\frac{1}{2}$-design with parameters (8, 16, 5, 10; 25, 35). Then there is a directed strongly regular graph with parameters (48, 30, 25, 15, 25) according to Theorem 2.1. However, it is known that there is no DSRG ($48m$, $30m$, $25m$, $15m$, $25m$) for any positive integer m by Jørgensen [5]. So, although the parameter set $(\mathbf{v}, \mathbf{b}, \mathbf{k}, \mathbf{r}; a, b) = (8, 16, 5, 10; 25, 35)$ satisfies all the necessary conditions imposed in Section 3.3 of [6], there is no $1\frac{1}{2}$-design with these parameters.

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